Calculation of Bounds on the Ergodic Capacity for Fading Channels with Dependency Uncertainty

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Abstract—**Modern applications of wireless communication systems often have strict performance requirements. Due to its random nature, channel fading is one of the most limiting factors to provide such guarantees. Even if knowledge about the statistics of the individual links to each antenna is available, there usually are additional uncertainties, e.g., imperfect channel-state information (CSI) or dependency uncertainty between multiple fading links. In this work, we consider the latter and show that a rearrangement algorithm can be applied to calculate the minimum and maximum ergodic capacity for fast fading channels when only the marginal fading distributions are known but the joint distribution is unknown. The algorithm can be used for any number of channels and supports arbitrary marginal distributions. The results are useful for communication system designers, e.g., using the worst-case ergodic capacity for robust system design.**

Index Terms—**Diversity methods, Fading channels, Joint distributions, Ergodic capacity, Rearrangement algorithm.**

I. Introduction

Current and future applications involving wireless data transmission have strict reliability constraints [1]. Unfortunately, there exist a lot of uncertain parameters due to the nature of wireless transmission channels. It is therefore of interest to design communication systems robustly such that their performance, e.g., outage probability or average performance, can be guaranteed even in the worst case. Often in literature, an uncertainty about the channel estimation (imperfect channel-state information (CSI)) is considered [2], [3]. In our work, we will instead consider an uncertainty about the joint distribution of the fading channels. This uncertainty stems from the fact that it is easy to measure the marginal fading distributions at multiple antenna positions individually. However, their joint distribution is usually unknown. It is a common assumption that they are statistically independent. However, measurements have shown that they can behave according to a more complicated dependency structure [4]. The mathematical tools to describe joint distributions are usually taken from copula theory [5]. An introduction on how they can be applied in communications can be found in [6], [7]. They have already been applied to derive the performance region of communication systems with dependency uncertainty in [8] and [9]. It is further shown in [10] that there exist joint slow-fading distributions for which a communication is possible at a positive rate and without any outages. In [11], the authors exploit copulas to derive capacity of correlated MIMO Nakagami-m channels. The recent work [12] also considers copula-based analysis of communication systems.

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In this work, we will consider the ergodic capacity and provide bounds for fast fading channels with dependency uncertainty. The transmitter needs to choose a modulation and coding scheme (MSC) according to the achievable rate. However, the rate is not known precisely, if the joint distribution is unknown. The results of this work are therefore useful for communication system designers, e.g., the minimum ergodic capacity serves as the worst-case scenario for robust system design. Since the general case of n inhomogeneous marginal distributions is still an open problem for $n > 2$, we show how the rearrangement algorithm (RA) from [13] can be successfully applied to numerically calculate the performance bounds. In addition, it can be quite cumbersome to obtain analytical solutions, even for the case of a known dependency structure [14], [15]. The RA is able to efficiently calculate the bounds, even when n is in the hundreds [13].

The presented bounds are the minimum and maximum ergodic capacities with respect to all joint fading distributions with given marginals. This therefore includes correlation models, e.g., as used in [16], [17]. The performance of such specific dependency structures lies somewhere between the upper and lower bounds that we derive in this work.

The rest of the paper is organized as follows. In Section II, we will present our system model and problem formulation, and briefly introduce the RA. In Section III, we will first show that the RA can find bounds on the ergodic capacity and we will then evaluate two different examples of fading distributions to 1) demonstrate the impact of the joint distribution on the ergodic performance, and 2) illustrate the advantages of applying the RA. Finally, Section IV concludes the paper.

Notation: Throughout this work, we use the following notation. Random variables are denoted in capital boldface letters, e.g., $\boldsymbol{X}$, and their realizations in small letters, e.g., x. We will use F and f for a probability distribution and its density, respectively. The expectation is denoted by $\mathbb{E}$ and the probability of an event by Pr. It is assumed that all considered distributions are continuous. The uniform distribution on the interval $[a, b]$ is denoted as $\mathcal{U}[a, b]$. The normal distribution with mean μ and variance σ^2 is written as $\mathcal{N}(\mu, \sigma^2)$; the log-normal distribution, derived from it, is written as $\mathcal{LN}(\mu, \sigma^2)$.

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The real numbers are denoted by $\mathbb{R}$. Logarithms, if not stated otherwise, are assumed to be with respect to the natural base.

II. Preliminaries

In this section, we will first present the system model that we consider in this work and state the problem formulation. Next, we will provide some mathematical background on bounds for the expected value of a certain class of functions on random variables. Finally, we will introduce the RA from [18] which was originally developed for quantitative risk management [19] and will be adapted in the following sections to compute bounds on the ergodic capacity.

A. System Model and Problem Formulation

Throughout this work, we will consider a single-input multiple-output (SIMO) fast fading channel with n receive antennas. The receiver has perfect CSI, e.g., by the use of pilot-signals, while the transmitter only has statistical CSI. The received signal at antenna i at time k is given by

$$\boldsymbol{Y}_i[k] = \boldsymbol{H}_i[k]\boldsymbol{M}[k] + \boldsymbol{W}_i[k] \,, \tag{1}$$

where $\boldsymbol{H}_i$ represents the fading coefficient, $\boldsymbol{M}$ the transmitted signal, and $\boldsymbol{W}_i \sim \mathcal{N}(0, \sigma^2)$ is additive white Gaussian noise (AWGN) with noise power σ^2. The transmission is subject to an average power constraint P which gives the signal-to-noise ratio (SNR) as $\rho = P/\sigma^2$. We assume that the fading process $\{\boldsymbol{H}_i\}$ is stationary and ergodic. In the following, we will omit the time index k.

We assume maximum ratio combining (MRC) at the receiver which gives the instantaneous channel capacity as [20]

$$C_{\text{inst}} = \log_2\left(1 + \rho\sum_{i=1}^{n}\boldsymbol{X}_i\right) , \tag{2}$$

where we introduce the shorthand $\boldsymbol{X}_i = |\boldsymbol{H}_i|^2$.

Since we consider fast fading channels, our quantity of interest is the ergodic capacity [20]

$$C = \mathbb{E}_{(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n)\sim F_{\boldsymbol{X}}}\left[C_{\text{inst}}\right] , \tag{3}$$

where the expectation is taken with respect to the joint distribution $F_{\boldsymbol{X}}$ of the vector $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$.

The transmitter needs to select a MSC according to the achievable rate. However, this rate is not known if the joint distribution is unknown. Robust design based on the worst case joint distribution guarantees a reliable communication. For such a design, we need the minimum of C. On the other hand, the best case, i.e., the maximum of C, can be used as a benchmark to assess the performance of a communication system. With these considerations we get our problem formulation as follows.

What are the best and worst case ergodic capacities, if only the marginal distributions $F_{\boldsymbol{X}_i}$ are known at the transmitter but the joint distribution $F_{\boldsymbol{X}}$ between the fading gains $\boldsymbol{X}_i$ is unknown? The problem can be formulated mathematically as the following optimization problems

$$C_{\min} = \inf_{F_{\boldsymbol{X}}:\boldsymbol{X}_i\sim F_{\boldsymbol{X}_i}} C \quad \text{and} \quad C_{\max} = \sup_{F_{\boldsymbol{X}}:\boldsymbol{X}_i\sim F_{\boldsymbol{X}_i}} C \,. \tag{4}$$

Remark 1. Another application is the fast fading multiple access channel (MAC) with fixed marginal distributions and unknown joint distribution. In this case, the bounds are for the sum-rate performance.

B. Bounds on the Expected Value of Supermodular Functions

In the following, we will present some mathematical background on minimum and maximum of the expected value of supermodular functions of random variables. In Section III, we will show that the ergodic capacity belongs to this class of functions. We can therefore apply the existing results that we review and adopt in the following to our problem.

First, we state the definition of a supermodular function.

Definition 1 (Supermodular Function [21, Def. 2.3.1])**.** Suppose X is a subset in $\mathbb{R}^n$ and a function $f : X \to \mathbb{R}$. The function f is supermodular on a set X, if for any $x, x' \in X$,

$$f(x) + f(x') \leq f(x \vee x') + f(x \wedge x') \,, \tag{5}$$

whenever $x \wedge x', x \vee x' \in X$, and where $x \vee x'$ and $x \wedge x'$ denote the component-wise maximum and minimum, respectively.

Remark 2. If the function f is supermodular, we call $-f$ submodular.

As stated in our problem formulation (4), we are interested in bounding the expected value of a function of random variables $\boldsymbol{X}_i$. It is well known that the maximum is attained for comonotonic $\boldsymbol{X}_i$, if the function is supermodular [22, Thm. 5.2.3]. Note that this corresponds to the minimum for submodular functions [6, Rem. 2].

On the other hand, it is more complicated to find the minimum on the expected value of supermodular functions over all joint distributions. If $n = 2$, the minimum is attained by countermonotonic random variables [23, Thm. 3.1.2], i.e., $\boldsymbol{X}_1 = F^{-1}_{\boldsymbol{X}_1}(\boldsymbol{U})$ and $\boldsymbol{X}_2 = F^{-1}_{\boldsymbol{X}_2}(1 - \boldsymbol{U})$ with $\boldsymbol{U} \sim \mathcal{U}[0, 1]$. However, no analytical solution is known for the general n-dimensional case. For the special case that the function is a convex function of the sum of the random variables and that all random variables have the same marginal distribution with a monotone density, the solution is derived in [24]. However, it requires solving integrals that might not have a closed-form solution.

It is therefore of interest to have an efficient way to determine the minimum in the general case. An algorithm to calculate this numerically is the RA [13], which will be introduced in the following.

C. Rearrangement Algorithm

The RA that we will use in the remaining part of this work was first introduced in [18] to find sharp bounds on the joint probability of random variables with an unspecified dependency. It was later extended to calculate bounds on the expected value of supermodular functions of random variables [13].

In this section, we need the following additional notation. Given a matrix $A \in \mathbb{R}^{N\times n}$, we obtain the matrix $A_{(-j)} \in$

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$\mathbb{R}^{N\times n-1}$ by deleting the j-th column $A_{(j)}$ from A. The row-wise application of a function ψ to a matrix A is denoted as

$$\psi(A) = \begin{pmatrix} \psi\left(a_{1,1}, a_{1,2}, \dots, a_{1,n}\right) \\ \vdots \\ \psi\left(a_{N,1}, a_{N,2}, \dots, a_{N,n}\right) \end{pmatrix}.$$

Recall that the goal is to find a joint distribution of $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ that minimizes the expected value

$$\min_{F_{\boldsymbol{X}}:\boldsymbol{X}_i\sim F_{\boldsymbol{X}_i}} \mathbb{E}_{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_n)\sim F_{\boldsymbol{X}}} \left[\psi(\boldsymbol{X}_1, \dots, \boldsymbol{X}_n)\right] \tag{6}$$

of a supermodular function ψ. The idea of the RA is to reformulate this problem as a problem of matrix rearrangements. We start by considering a matrix $A \in \mathbb{R}^{N\times n}$ where the j-th column $A_{(j)}$ represents the random variable $\boldsymbol{X}_j$. The entries $a_{i,j}$ of the j-th column are constructed by quantizing the continuous distribution $F_{\boldsymbol{X}_j}$ with N quantization steps. This can be done in the following two ways [13]

$$\underline{a}_{i,j} = F^{-1}_{\boldsymbol{X}_j}\left(\frac{i-1}{N}\right) \quad \text{and} \quad \overline{a}_{i,j} = F^{-1}_{\boldsymbol{X}_j}\left(\frac{i}{N}\right), 1 \le i \le N,$$

which yield the matrices $\underline{A}$ and $\overline{A}$, respectively. These matrices are then rearranged iteratively such that in each step the j-th column $A_{(j)}$ is oppositely ordered to $\psi(A_{(-j)})$. The step is repeated for each column until convergence. The opposite rearrangement makes the random variables represented by $A_{(j)}$ and $\psi(A_{(-j)})$ countermonotonic. The algorithm therefore produces a form of n-dimensional countermonotonicity between the $\boldsymbol{X}_j$. The optimal dependency structure for some special cases is investigated in more detail in [24].

The RA finally yields the two matrices $\underline{A}^\star$ and $\overline{A}^\star$ as results after the rearrangement of $\underline{A}$ and $\overline{A}$, respectively. The minimum of the expected value in (6) is then bounded by

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{N} \psi\left(\underline{a^\star_{i,1}}, \underline{a^\star_{i,2}}, \dots, \underline{a^\star_{i,n}}\right) &\le \min_{F_{\boldsymbol{X}}} \mathbb{E}\left[\psi(\boldsymbol{X}_1, \dots, \boldsymbol{X}_n)\right] \\ &\le \frac{1}{N}\sum_{i=1}^{N} \psi\left(\overline{a^\star_{i,1}}, \overline{a^\star_{i,2}}, \dots, \overline{a^\star_{i,n}}\right). \end{aligned} \tag{7}$$

For the mathematical details, we refer the reader to [13], [18].

This means that the RA yields a range in which the exact value of the minimum expected value lies. This size of the gap between upper and lower bound is controlled by the number of quantization steps N and vanishes for $N \to \infty$. As a byproduct, the algorithm also determines a quantized version of the optimal joint distribution of the $\boldsymbol{X}_i$, which is given as the columns of the rearranged matrices $\underline{A}^\star$ and $\overline{A}^\star$. This provides useful insight on how to achieve the maximum ergodic capacity, if a proactive design of the dependency structure between the fading channels becomes possible, e.g., with upcoming technologies like reconfigurable intelligent surface (RIS) [25].

An implementation of the RA is available in the R library `qrmtools` [26]. A detailed discussion of the implementation can be found in [27].

Remark 3. In the following Section III, we will show that the RA can be used to calculate the maximum ergodic capacity $C_{\max}$. Based on the above description of the algorithm, we can see that we actually obtain lower and upper bounds of the exact value as

$$\underline{C_{\max}} \le \max_{F_{\boldsymbol{X}}:\boldsymbol{X}_i\sim F_{\boldsymbol{X}_i}} C = C_{\max} \le \overline{C_{\max}}. \tag{8}$$

III. Bounds on the Ergodic Capacity

In this section, we will first show that we can use the RA to find bounds on the ergodic capacity from (3). Next, we will evaluate them explicitly for different fading distributions and compare them to analytical solutions and the independent case.

A. Application of the RA to Bound the Ergodic Capacity

In order to being able to apply the RA from [13], we need to show that our target function is supermodular. Recall that $-f$ is supermodular, if f is submodular.

Corollary 1. *The instantaneous channel capacity defined in* (2) *is a submodular function.*

Proof. Since the function is a concave function of the sum of the components of x, it follows from [21, Thm. 2.3.6(a)] that $-f$ is supermodular. Hence, f is submodular. □

From Corollary 1, we know that we can apply the RA to bound the ergodic capacity from (3).

The source code to recreate all results in this work can be found at [28]. The implementation of the RA is adapted from the `qrmtools` R library [26], [27].

B. Homogeneous Rayleigh Fading

As a first example, we will take a look at the case of homogeneous Rayleigh fading. In this scenario, all channel coefficients $\boldsymbol{H_i}$ are distributed according to the same Rayleigh distributions. Therefore, we get that $\boldsymbol{X}_i \sim \exp(1)$ for all $i = 1, \dots, n$. Note that $\boldsymbol{X}_i$ has a monotone density in this case and we can therefore apply the theory of [24] to find exact solutions of the supremum in (4).

1) Minimum Ergodic Capacity: As stated in Section II-B, the minimum ergodic capacity is attained for comonotonic $\boldsymbol{X}_i$.

In this case, the lower bound on the ergodic capacity is calculated as

$$\begin{aligned} \min_{F_{\boldsymbol{X}}:\boldsymbol{X}_i\sim\exp(1)} C &= \mathbb{E}_{\boldsymbol{X}_1\sim\exp(1)}\left[\log_2\left(1+\rho n \boldsymbol{X}_1\right)\right] & (9)\\ &= \int_0^\infty \log_2\left(1+\rho n x\right)\exp(-x)\mathrm{d}x & (10)\\ &= -\frac{1}{\log(2)}\exp\left(\frac{1}{n\rho}\right)\mathrm{Ei}\left(\frac{-1}{n\rho}\right), & (11) \end{aligned}$$

where Ei is the exponential integral [29, Chap. 5].

It is also possible to use the RA to determine the minimum ergodic capacity. In this case, the quantization is done as described in Section II-C. However, no rearrangement is necessary as the columns are arranged comonotonically by default. As described in (8), the resulting $\underline{C_{\min}}$ and $\overline{C_{\min}}$ are bounds on the exact value of the minimum ergodic capacity

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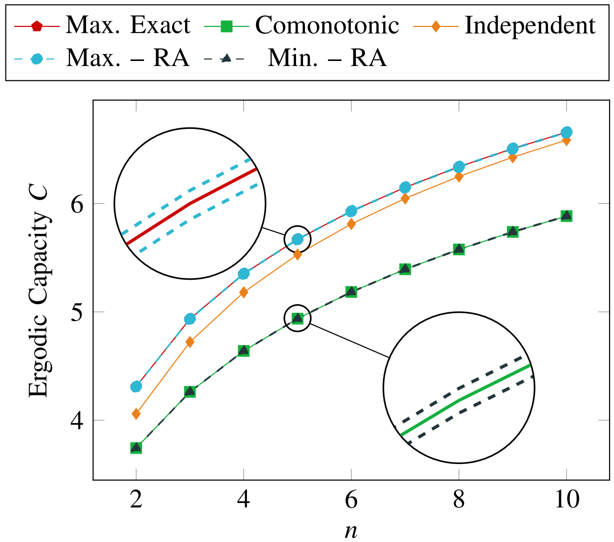

Figure 1. Bounds on the minimal and maximal ergodic capacity for n dependent Rayleigh fading channels derived by the RA. The comonotonic case corresponds the exact values of the minimum. The parameters are $\rho = 10\,\mathrm{dB}$ and $N = 1000$.

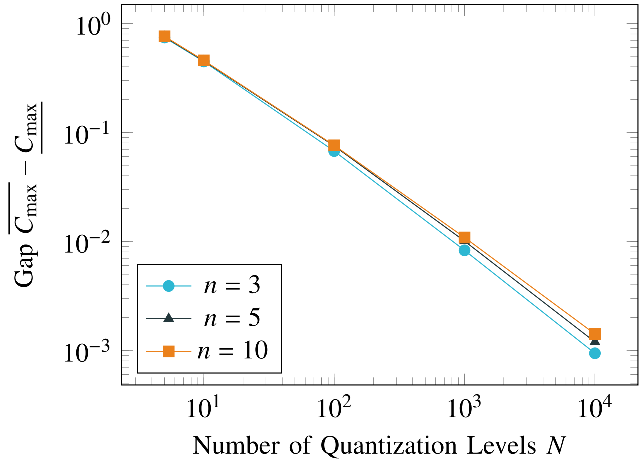

Figure 2. Gap between the bounds on the maximum ergodic capacity for n dependent Rayleigh fading channels calculated using the RA over the number of quantization steps N. The SNR is set to $\rho = 10\,\mathrm{dB}$.

$C_{\min}$ that is stated in (11). An example of both the exact values and bounds derived by the RA is shown in Fig. 1.

2) Maximum Ergodic Capacity: A general solution for the maximum of the expected value of a submodular function (or equivalently the minimum of a supermodular function) remains an open problem. However, for the special case that the supermodular function is a convex function of the sum of n random variables with the same marginal distribution with a monotone density, an analytical solution is presented in [24]. For this example, we have that all $\boldsymbol{X}_i$ are distributed according to the same exponential distribution. Since the exponential distribution has a monotone density, we can apply the theory of [24] to find the exact solution to

$$C_{\max} = \max_{F_{\boldsymbol{X}}:\boldsymbol{X}_i \sim \exp(1)} C\,.$$

However, this requires calculating integrals which have no closed-form solution. More details can be found in [8], [24]. The results, that are presented in the following, are therefore determined numerically. The source code for all calculations can be found in interactive notebooks at [28]. An example of both the exact values and bounds derived by the algorithm is shown in Fig. 1.

As stated in Section II-C, the RA yields a range $(\underline{C}_{\max}, \overline{C}_{\max})$ in which the exact value of the maximum ergodic capacity $C_{\max}$ lies. The gap $\overline{C}_{\max} - \underline{C}_{\max}$ reduces with the number of quantization levels N. In Fig. 2, the gap $\overline{C}_{\max} - \underline{C}_{\max}$ is shown over different N for multiple numbers of fading links n. It can be seen that the gap decreases approximately with a diversity of 1 with the number of used quantization levels N.

3) Independent Case: For comparison, we also calculate the ergodic capacity in the case that all $\boldsymbol{X}_i$ are independent and identically distributed (i.i.d.). In this case, the sum $\sum_{i=1}^{n} \boldsymbol{X}_i$ is Gamma-distributed [30, Chap. 17]. The ergodic capacity is calculated as

$$C_{\mathrm{ind}} = \mathbb{E}_{\boldsymbol{S}\sim\Gamma(n,1)}\left[\log_2\left(1+\rho\boldsymbol{S}\right)\right] \tag{12}$$

$$= \exp\left(\frac{1}{\rho}\right)\log_2(\mathrm{e})\sum_{i=1}^{n-1} E_{i+1}\left(\frac{1}{\rho}\right)\,, \tag{13}$$

where (13) is taken from [31, Eq. (19)]. The calculations can also be found in an interactive notebook at [28].

4) Summary: Figure 1 shows the best case, worst case and i.i.d. case for homogeneous Rayleigh fading. The SNR is set to $10\,\mathrm{dB}$ and the number of quantization levels in the RA is $N = 1000$. The worst case, i.e., minimum ergodic capacity, is given exactly by the comonotonic case in (11). For comparison, also the results from the RA are shown. Recall that the algorithm yields both upper and lower bounds which lie around the exact value as expected. Similarly, we can determine the exact value of the maximum ergodic capacity using the results from [24, Thm. 3.5]. Note that this only works for this particular example where we have homogeneous fading links with a monotone density. As expected, the values for the i.i.d. case lie between the best and worst case. However, it should be noted that the gap between the maximum and the i.i.d. curve decreases for increasing n. At this point, we are not able to prove analytically that the gap gets arbitrarily small for $n \to \infty$.

C. Arbitrary Fading Distributions

As previously mentioned, the minimum ergodic capacity $C_{\min}$ is achieved for comonotonic fading gains $\boldsymbol{X}_i$. This can be easily calculated for arbitrary fading distributions by setting $\boldsymbol{X}_i = F_{\boldsymbol{X}_i}^{-1}(\boldsymbol{U})$ with $\boldsymbol{U} \sim \mathcal{U}[0,1]$ [5]. On the other hand, there currently is no analytical solution known for the general problem of finding the maximum ergodic capacity $C_{\max}$. The theory from [24] only works for the special case of homogeneous distributions with monotone densities. Many

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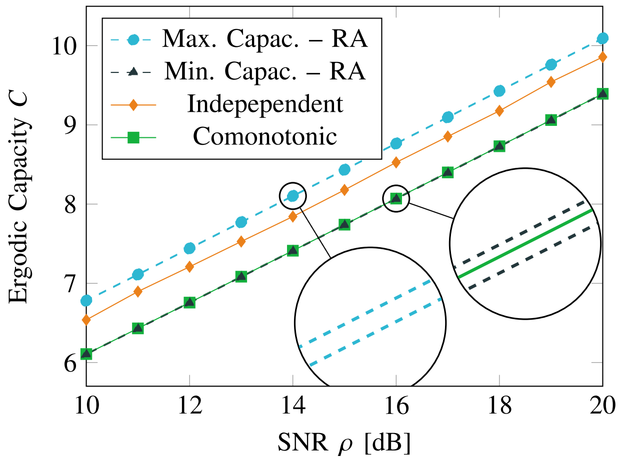

Figure 3. Ergodic capacities for $n = 3$ fading channels with $\boldsymbol{X}_1 \sim \exp(1/4)$, $\boldsymbol{X}_2 \sim \mathcal{LN}(0, 2\log(4))$, and $\boldsymbol{X}_3 \sim \chi^2_4$. The curves labeled with "RA" are determined by the rearrangement algorithm. The number of quantization levels for the RA is set to $N = 1000$.

common fading distributions, e.g., log-normal and Nakagami-m, do *not* fulfill this property. For these cases, the RA can still be applied and gives us an efficient option to calculate $C_{\max}$.

In order to demonstrate this, we consider the following example. We have $n = 3$ fading links, where the first channel gain is exponentially distributed, i.e., $\boldsymbol{X}_1 \sim \exp(\lambda)$; the second is log-normally distributed with $\boldsymbol{X}_2 \sim \mathcal{LN}(0, s^2)$; and the third channel gain is distributed according to a Chi-square distribution, i.e., $\boldsymbol{X}_3 \sim \chi^2_k$. We set the degrees of freedom of $\boldsymbol{X}_3$ to $k = 4$. In order to have the same expected value for all $\boldsymbol{X}_i$, we set the other parameters to $\lambda = 1/4$ and $s^2 = 2\log(4)$.

1) Minimum Ergodic Capacity: The minimum ergodic capacity $C_{\min}$ is given by comonotonic $\boldsymbol{X}_i$, i.e., $\boldsymbol{X}_i = F^{-1}_{\boldsymbol{X}_i}(\boldsymbol{U})$ with $\boldsymbol{U} \sim \mathcal{U}[0, 1]$. This yields the general expression

$$C_{\min} = \mathbb{E}_{\boldsymbol{U}\sim\mathcal{U}[0,1]}\left[\log_2\left(1 + \rho\sum_{i=1}^{n} F^{-1}_{\boldsymbol{X}_i}(\boldsymbol{U})\right)\right]. \tag{14}$$

For our example, the evaluated minimum capacities over different SNRs are shown in Fig. 3. The calculation of (14) is done via numerical integration. The source code can be found at [28].

Besides the numerical integration, we can again use the RA to determine a range $(\underline{C}_{\min}, \overline{C}_{\min})$ in which the exact minimum ergodic capacity $C_{\min}$ lies. The curves are also shown in Fig. 3. As in the previous example, the gap between them is small and we can therefore use the RA to determine the exact value with only a small error.

2) Maximum Ergodic Capacity: Since the marginals are not homogeneous, we cannot leverage the results from [24] to calculate the exact values of the maximum ergodic capacity $C_{\max}$. However, we can still apply the RA to determine the range $(\underline{C}_{\max}, \overline{C}_{\max})$ in which $C_{\max}$ lies. For the evaluation shown in Fig. 3, we used $N = 1000$ quantization levels. The gap between $\overline{C}_{\max}$ and $\underline{C}_{\max}$ in this case is around 0.01. This means that we can determine the maximum ergodic capacity for this example with three heterogeneous marginal distributions up to an accuracy of 0.01.

For a smaller number of quantization levels the gap increases, but is still in a reasonable area. For the considered example, the gap is around 0.05 for $N = 200$ and around 0.3 for $N = 20$. This behavior and the exact values can be observed in an interactive notebook at [28].

3) Independent Case: For comparison, we also show the ergodic capacities for the case that $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, and $\boldsymbol{X}_3$ are independent. The values presented in Fig. 3 are found by Monte Carlo simulations with 10^4 samples. The source code can be found at [28]. Similarly to the previous example of homogeneous Rayleigh fading, the independent case is closer to the best case than to the worst case.

4) Summary: This example with three heterogeneous marginal distributions shows the strength of using the RA to determine minimum and maximum of the ergodic capacity for fading channels with dependency uncertainty. The presented results show that the gap from the RA is reasonably small, even for a small number of quantization levels. The algorithm is therefore a valid choice to efficiently calculate bounds on the ergodic capacity for fading channels with an unknown dependency structure.

From Fig. 3, it can be seen that there is a significant gap between the best case and worst case ergodic capacities. This shows that the joint distribution between the multiple fading channels has a strong impact on the performance of wireless communication systems. It is therefore important to consider the results of this work, e.g., the worst case bound for robust system design, if there is an uncertainty about the joint distribution. As clearly visible in both Fig. 1 and Fig. 3, the common assumption of independent channels has a significant gap to the lower bound. Assuming independence would therefore be too optimistic for robust system design and could lead to violations of the performance requirements.

IV. Conclusion

In this work, we presented a way to calculate the maximum and minimum ergodic capacity for fast fading channels with arbitrary fixed marginals and an arbitrary dependency structure between them. We adapt the RA from [13] to our problem setup. We first showed that it can be applied to the problem of ergodic capacities. Next, two examples with practical fading distributions are considered. First the case of homogeneous Rayleigh fading. For this specific example, the exact solutions to the maximum and minimum ergodic capacity are known and could be used to evaluate the accuracy of the RA. In the second example, three different fading distributions are considered. Since no exact solution for the maximum ergodic capacity is known in this heterogeneous case, the RA is especially useful in this scenario.

The results of this work are useful for communication system designers when there is an uncertainty about the joint distribution of the fading gains. First, the minimum ergodic capacity can be used as a worst-case scenario for robust design. Second, the maximum ergodic capacity can be used as a

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benchmark to assess how good the system performs compared to the best possible. With technologies like RIS [25] or smart relaying [32], it might be possible to actively control the dependency structure between different fading links. In this case, the optimal joint distribution found by the RA might help as a design guideline to improve the system's performance.

In future work, we will extend this framework to other diversity combining schemes. In particular, it will be interesting for selection combining in the case that multiple radio access technologies are used. Another advantage of the RA is that it can also deal with measured distributions. The application to real channel measurements will also be considered in future work.

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