Abstract—Unreliable fading wireless channels are the main challenge for strict performance guarantees in mobile communications. Diversity schemes including massive number of antennas, huge spectrum bands and multi-connectivity links are applied to improve the outage performance. The success of these approaches relies heavily on the joint distribution of the underlying fading channels. In this work, we consider the $\epsilon$-outage capacity of slowly fading wireless diversity channels and provide lower and upper bounds for fixed marginal distributions of the individual channels. This answers the question about the best and worst case outage probability achievable over $n$ fading channels with a given distribution, e.g., Rayleigh fading, but not necessarily statistically independent. Interestingly, the best-case joint distribution enables achieving a zero-outage capacity greater than zero without channel state information at the transmitter for $n \geq 2$. All results are specialized to Rayleigh fading and compared to the standard assumption of independent and identically distributed fading component channels. The results show a significant impact of the joint distribution and the gap between worst- and best-case can be arbitrarily large.

Index Terms—Diversity methods, Fading channels, Network reliability, Joint distributions, Outage capacity.

I. INTRODUCTION

With advances in communication technology, more critical applications start to rely on wireless transmission, e.g., car-to-car communication and medical applications [1]. These areas make high demands on reliability. Therefore, research started to focus on problems like ultra-reliable communications where very low error-rates of less than $10^{-3}$ are required [2]. In order to understand the trade-offs and efficient operating points for ultra-reliable communications, [2] develops a framework by listing enabling technologies and methods as well as their application in the use cases 1) enhanced Mobile Broadband (eMBB), 2) massive Machine-Type Communication (mMTC), and 3) ultra-reliable low latency communication (URLLC) [3].

In mobile wireless settings, the communication channel is usually modeled as a (slow) fading channel [4]. In this type of channel, one cannot transmit code words with an arbitrarily small error probability even for an infinite blocklength [5, Chap. 5]. Therefore, the $\epsilon$-outage capacity is used as a performance metric. It is defined as the largest transmission rate for which the outage probability is still less than $\epsilon$ [5].

In this reliable communication over fading channels scenario, we have the situation where we have collected many fading channel gain $X_i$ measurements at $n$ specific points in space and/or frequency, from which we can deduce the marginal distribution $F_i$ of the fading channels at these points. The question now arises: What can we say about the achievable outage performance $P(S < r)$ for $r > 0$ and $S = X_1 + \ldots + X_n$, if we employ a receive diversity system at these points and, e.g., perform coherent combining?

From the theoretical side, the impact of the joint distribution on different performance metrics is studied in the context of entropy in [6] and for performance bounds in [7]. The problem of optimizing the joint distribution given marginal distributions can be formulated by the copula approach [8]. Recently, [9] exploits copulas and coupling to derive stochastic orders for multi-user fading channels to characterize ergodic capacity regions of multi-user channels [10].

However, most of the previous work on performance of diversity systems in fading wireless channels only considers the case of independent or positively correlated fading processes [11]. In this work, we will partly answer the question above and provide bounds for cases which are relevant for the fading wireless channels.

We provide both upper and lower bounds on the $\epsilon$-outage capacity for fading wireless channels with monotone marginal densities allowing dependent fading processes. The main contributions are the following. At first, we provide lower and upper bounds for fading channels where all fading coefficients $|h_i|^2 \sim F$ follow the same distribution with a monotone density in Theorem 1. The proofs of these bounds can be found in [12]. Second, these derived results are applied to the typical Rayleigh fading model and the bounds are stated explicitly.

The paper is organized as follows. In Section II, preliminaries and the system model are introduced. The application of these results for the special case of Rayleigh fading channels is given in Section III including numerical assessment of the impact of different system parameters on the bounds and the state of the art, mainly the independent and identically distributed (i.i.d.) case. Section IV concludes the paper.

Notation

Throughout this work, we will use $F$ and $f$ for a probability distribution and its density, respectively. The expected value is denoted by $\mathbb{E}$ and the probability by $\mathbb{P}$. The function $G = F^{-1}$ denotes the inverse of $F$. It is assumed that all considered
distributions are continuous. Real-valued Gaussian and complex circularly symmetric Gaussian random variables with mean \( \mu \) and covariance matrix \( \Sigma \) are denoted as \( \mathcal{N}(\mu, \Sigma) \) and \( \mathcal{CN}(\mu, \Sigma) \), respectively. The uniform distribution on the interval \([a, b]\) is denoted as \( \mathcal{U}(a, b) \).

II. PRELIMINARIES

A. System Model

Throughout this work, we consider the complex flat fading channel [5, Ch. 5.4] with \( n \) receive dimensions. These could be \( n \) antennas placed spatially or \( n \) distributed receivers like in multi-connectivity, or \( n \) time or frequency instances, over which the same symbol \( x \) is transmitted. The received signal at discrete time \( k \) is given by the vector \( y \) with \( n \) components as

\[
y[k] = h[k]x[k] + w[k],
\]

where \( h[k] = [h_1[k], \ldots, h_n[k]] \) represents the fading channel and \( w[k] \sim \mathcal{CN}(0, N_0) \) is i.i.d. complex Gaussian noise with zero mean and variance \( N_0 \). In the following, we will drop the time instance \( k \) since we assume that \( h \) is constant for the code word length and the corresponding rate expressions are achievable.

If the transmitter has no channel-state information (CSI), and the receiver has perfect CSI on the slow-fading channel, then the definition of the e-outage capacity \( R_e \) for this channel model is given as [5]

\[
R_e = \sup_{R \geq 0} \left\{ R \in \mathbb{R} : P \left( \sum_{i=1}^{n} |h_i|^2 < \frac{2R - 1}{\rho} \right) < \varepsilon \right\}
\]

for a certain signal-to-noise ratio (SNR) \( \rho \). The SNR of the channel is defined as \( P/N_0 \) where \( P \) is the transmit power. The e-outage capacity is the transmission rate which results in an outage probability of at most \( \varepsilon \) if the next channel realization is used.

One extreme case of e-outage capacity is the zero-outage capacity [13] or delay-limited capacity [14], where \( \varepsilon \) is set to zero. This means the rate \( R^0 \) is achievable for all channel realizations. It is well known that without channel-state information at the transmitter (CSI-T), this is not possible (at least for i.i.d. fading), while with perfect CSI-T it can be achieved with a long-term power constraint and channel inversion [15].

B. Bounds on the Outage Performance or Risk

In many different areas, the sum of multiple random variables with (unspecified) dependency structure is of interest. An example is the total risk in risk management [16]. Since the dependency between the different variables is usually unknown, bounds like the worst and the best case are of particular interest. In [17], the authors provide bounds on the probability of the sum of dependent variables \( X_i \sim F_i \) with monotone marginals. For the homogeneous case of \( F_1 = \cdots = F_n = F \) with monotone density, the authors provide a formulation of a function \( \phi \) such that

\[
\phi^{-1}(s) = \inf_{F_{X_1}, \ldots, X_n} \left\{ \mathbb{P}\left( \sum_{i=1}^{n} X_i < s \right) : X_i \sim F \right\}.
\]

In order to do this, they prove the existence of an optimal coupling between the dependent variables by a copula \( Q_F^c \). The dependency structure is constructed in such a way that the sum of the random variables \( X_i = G(U_i) \) is a constant when any of them is in a certain interval, i.e., \( U_i \in (c_n, 1 - (n - 1)c_n) \), where \( U_i \) is a uniformly distributed random variable on the interval \([0, 1]\) [18], i.e., \( U_i \sim \mathcal{U}[0, 1] \). For \( c_n = 0 \), this sum simplifies to \( n\mathbb{E}[X_1] \). In case of \( c_n > 0 \), this constant is equal to the conditional expectation, which is given by the function \( H \).

For decreasing densities, \( c_n \) and \( H \) are defined as [17]

\[
c_n(a) = \min \left\{ c \in [0, \frac{1-a}{n}] : \int_{c}^{1-a/n} H_d(t)dt \geq \left( \frac{1-a}{n} - c \right) H_d(c) \right\},
\]

and

\[
H_d(x) = (n-1)G(a + (n-1)x) + G(1-x),
\]

with \( a \in [0, 1] \).

For increasing densities, \( c_n \) and \( H \) are given as

\[
c_n(a) = \min \left\{ c \in [0, \frac{1-a}{n}] : \int_{c}^{1-a/n} H_d(t)dt \leq \left( \frac{1-a}{n} - c \right) H_d(c) \right\},
\]

and

\[
H_d(x) = G(a + x) + (n-1)G(1-(n-1)x),
\]

with \( a \in [0, 1] \).

Combining (4)-(7) into one function \( \phi \), gives the following definitions for decreasing densities

\[
\phi(a) = \begin{cases} 
H_d(c_n(a)) & \text{if } c_n(a) > 0 \\
\mathbb{E}[X|X > G(a)] & \text{if } c_n(a) = 0
\end{cases},
\]

and increasing densities

\[
\phi(a) = \begin{cases} 
H_d(0) & \text{if } c_n(a) > 0 \\
\mathbb{E}[X|X > G(a)] & \text{if } c_n(a) = 0
\end{cases}.
\]

Please refer to [17], [18] for the proofs and details.

C. Bounds on the e-Capacity for Monotone Marginals

The following theorems provide upper and lower bounds on the outage capacity for the scenario where all fading coefficients \( h_i \) have a monotone marginal distribution. The proofs to them can be found in [12].

In this work, we consider the case of identical marginals, i.e., \( |h_i|^2 \sim F, i = 1, \ldots, n \). For this case, upper and lower bounds on the e-outage capacity are stated in the following theorem.
Theorem 1 (Bounds on the \( \varepsilon \)-outage capacity for identical monotone marginals). The \( \varepsilon \)-capacity \( R^\varepsilon \) of \( n \) multi-connectivity fading links with monotone marginal distributions \( |h_i|^2 \sim F \) can be bounded by

\[
R^\varepsilon \leq R_0^\varepsilon \leq \overline{R}_0^\varepsilon.
\]  

(10)

The worst-case \( \varepsilon \)-capacity \( \overline{R}_0^\varepsilon \) is given as

\[
\overline{R}_0^\varepsilon(\rho) = \log_2 (1 + \rho \cdot \phi_-(1 - \varepsilon)),
\]

(11)

and the best-case \( \varepsilon \)-capacity \( R_0^\varepsilon \) is given as

\[
R_0^\varepsilon(\rho) = \log_2 (1 + \rho \cdot \phi(\varepsilon)),
\]

(12)

where \( \rho \) is the SNR and \( \phi \) is defined in Eq. (8) and (9).

Proof. The proof can be found in [12, Thm. 1].

In the previous statements, only perfect CSI at the receiver is assumed. We will now focus on the zero-outage capacity in the case of perfect CSI-T. As shown in [19], the zero-outage capacity \( R_0^0 \) with perfect CSI-T can be written as

\[
R_0^0 = \log_2 \left( 1 + \rho \left( \frac{1}{\sum_{i=1}^n |h_i|^2} \right)^{-1} \right).
\]

(13)

In the following, this is used as a basis to show how perfect CSI-T can improve the bounds on the zero-outage capacity.

Theorem 2. The best-case zero-outage capacity for fading channels with monotone decreasing densities and perfect CSI at the transmitter is given by

\[
\overline{R}_0^0 = \log_2 \left( 1 + \frac{\rho \cdot H(c_n)}{H(c_n) n \int_{c_n}^1 \frac{1}{H(1-x)} dx + 1 - nc_n} \right),
\]

(14)

where \( H \) and \( c_n \) are given by \( H_0 \) from (5) and \( c_n(0) \) from (4), respectively.

Proof. The proof can be found in [12, Lemma 4].

The worst case zero-outage capacity with perfect CSI-T is attained for comonotonic fading coefficients, which is shown in [12, Cor. 1].

### III. Rayleigh Fading

In this section, the special case of Rayleigh fading is considered. This channel model is commonly used for modeling mobile channels [4]. In this case, the amplitude of the fading coefficients \( |h_i|^2 \) follows a Rayleigh distribution and therefore \( |h_i|^2 \) is exponentially distributed with \( |h_i|^2 \sim \exp(\lambda) \) [20, Ch. 39].

The exponential distribution has a monotone decreasing density \( f(x) = \lambda \exp(-\lambda x) \) which allows us to apply Theorem 1 to determine lower and upper bounds for the outage capacity.

In this work, the case where all \( \lambda_i = 1 \) are the same is considered. In the following, the resulting bounds on the \( \varepsilon \)-outage capacity are shown along with the i.i.d. and comonotonic cases. The details, including Python code, for reproducing all of the following calculations in this section can be found at [21].

![Figure 1](https://doi.org/10.24355/dbbs.084-202301111337-0)

Figure 1. Computed values of \( c_n(a) \) according to (4) over \( a \) for different \( n \) with \( |h_i|^2 \sim \exp(1) \), \( i = 1, \ldots, n \).

#### A. Upper Bound

The upper bound \( \overline{R}_0^\varepsilon \) on the \( \varepsilon \)-outage capacity is calculated according to (12). In order to do this, \( \phi \) needs to be calculated according to (8), since \( |h_i|^2 \sim \exp(1) \) has a decreasing density. Hence, \( c_n(a) \) needs to be determined according to (4). In the case of exponentially distributed variables, this yields the following inequality

\[
(a - 1) \log \left( 1 - a - (n - 1) c_n \right) \geq n(a + cn - 1).
\]

(15)

However, this cannot be solved by a closed-form expression but rather numerically. Figure 1 shows the calculated values of \( c_n(a) \) for different \( n \) over \( a \). After determining \( c_n(a) \), they are used to calculate \( \phi \) and \( \overline{R}_0^\varepsilon \). For our considered case, \( \phi \) can be expressed as

\[
\phi(a) = \begin{cases} 
H_d(c_n(a)) & \text{if } c_n(a) > 0 \\
(n(1 - \log(1-a))) & \text{if } c_n(a) = 0.
\end{cases}
\]

(16)

For \( a \in [0,1) \), it can be seen from (15) that \( c_n(a) > 0 \). Therefore, \( \phi \) can be simplified to

\[
\phi(a) = H_d(c_n(a))
\]

(17)

\[
= -(n-1) \log(1-a-(n-1)c_n(a)) - \log(c_n(a)).
\]

(18)

This yields the following expression for the upper bound on the \( \varepsilon \)-outage capacity

\[
\overline{R}^\varepsilon(\rho) = \log_2 \left( 1 - \rho \cdot \left( (n-1) \log(1-\varepsilon-(n-1)c_n(\varepsilon)) + \log(c_n(\varepsilon)) \right) \right).
\]

(19)

An example for \( n = 5 \) and \( \rho = 5 \) dB is presented in Fig. 2 later. As described in [12, Prop. 2], the capacity in the best case is always positive and even the zero-outage capacity in the example has a value of around 4.06 bits per channel use. It increases with increasing \( \varepsilon \) and tends towards infinity for \( \varepsilon \to 1 \). Please note that the usual operating point is \( \varepsilon < 10^{-2} \) for URLLC.
B. Lower Bound

The lower bound $R^e_\text{L}$ on the $e$-outage capacity is calculated according to (11). The approach is very similar to the one of calculating the upper bound in the previous section. However, $\phi_-$ is required and all necessary steps have to be calculated for the random variables $-|h_i|^2 \sim \exp(1)$. The distribution of these variables has the increasing density $f(x) = \exp(x)$ on the support $(-\infty, 0]$. Using this gives the following inequality to be solved for $c$ according to (6)

$$\frac{n(a + nc - 1)}{1 + (n - 1)a} \leq \log(a + c) - \log(1 - (n - 1)c). \quad (20)$$

Depending on $c_n(a)$, $\phi_-$ can be expressed as

$$\phi_-(a) = \begin{cases} \log(a) & \text{if } c_n(a) > 0 \\ n \cdot \frac{a - a \log(a) - 1}{1 - a} & \text{if } c_n(a) = 0 \end{cases}. \quad (21)$$

Applying this to (11), gives the expression for the lower bound $R^e_\text{L}$.

**Remark 1.** It should be noted that in this case the solution $c_n(a) = 0$ occurs, if the following holds

$$\frac{n(a - 1)}{1 + (n - 1)a} \leq \log(a). \quad (22)$$

This inequality can be solved numerically and gives a minimum $a$, which is decreasing with $n$. As an example, the minimum $a$ for which the inequality holds is around 0.117 for $n = 3$ and $n = 10$, respectively. For the lower bound on the outage-capacity in Theorem 1, $\phi_-$ is evaluated at the argument $1 - e$. An interesting application of the outage-capacity is URLLC, where $e$ close to zero are considered. In this case, $1 - e$ is close to one and therefore the above inequality holds. Therefore, $c_n(1 - e)$ is zero in this case.

The calculated $R^e_\text{L}$ for a Rayleigh fading channel with $n = 5$ and $\rho = 5$ dB is shown in Fig. 2. One noticeable point is that the worst-case zero-outage capacity is zero. However, for positive $e$, it attains positive values and tends towards infinity for $e \to 1$. This is expected since a value of $e = 1$ implies that an arbitrarily large number of transmission errors is tolerated.

C. Comonotonic Coefficients

One special case are comonotonic coefficients, i.e., $|h_i|^2 \sim G(U)$ with $U \sim U[0, 1]$. In this case, the following holds

$$\mathbb{P} \left( \sum_{i=1}^{n} |h_i|^2 \leq \frac{2^R - 1}{\rho} \right) = \mathbb{P} \left( \left| h_1 \right|^2 \leq \frac{2^R - 1}{n \rho} \right) = F \left( \frac{2^R - 1}{n \rho} \right),$$

which combined with (2) yields the following expression for the $e$-outage capacity

$$R^e_{\text{comon}} = \log_2 \left( 1 - \rho \cdot n \log(1 - e) \right). \quad (23)$$

The outage capacity for this case is also shown for comparison in Fig. 2.

D. Independent Coefficients

In the case of i.i.d. fading coefficients $h_i$, the outage capacity can be given as a closed-form expression. Since the sum of independent exponentially distributed random variables with the same mean is Gamma-distributed [20, Ch. 17.2], the $e$-outage capacity in the independent case is given as

$$R^e_{\text{ind}} = \log_2 \left( 1 + \rho \frac{1}{n} \left( n \log(n/e) \right) \right), \quad (24)$$

where $P(a, x)$ is the regularized incomplete Gamma-function [22, Eq. 6.5.1] (which is the cumulative distribution function (CDF) of the Gamma-distribution [20]).

Figure 2 shows the $e$-outage capacity for Rayleigh fading coefficients with $n = 5$ and an SNR of $\rho = 5$ dB in the best case and worst case, along with the cases of i.i.d. and comonotonic coefficients.

As expected, the i.i.d. and comonotonic cases lie in between the other two and show a similar behavior. All curves increase with increasing $e$ and tends towards infinity for $e \to 1$. For small $e$, the outage capacity for comonotonic coefficients is smaller than for independent ones. However, there is an $e$ at which the curves meet and the capacity of the i.i.d. case becomes smaller. For the interesting operating region, the i.i.d. case performs better than the comonotonic case with significantly higher $e$-outage capacity.

Interestingly, all zero-outage capacities are zero except in the best case. This is due to the considered model of no CSI at the transmitter. In the case of perfect CSI at the transmitter, the zero-outage capacity in the i.i.d. case increases to a positive value, which is shown in [12].

Figure 3 shows the outage probability $e$ for $n = 5$ Rayleigh fading links at a fixed rate $R = 0.5$. The first thing to be noted is that the outage probability in the best case drops to zero at a certain SNR. This is expected since it was proven in [12, Prop. 2], that the zero-outage capacity for fading channels with decreasing density over the non-negative real numbers is
positive in the best case. Next, it can be seen that, similar to Fig. 2, the curves of the i.i.d. and comonotonic cases meet. For a low SNR, the outage probability is higher in the case of independent coefficients compared to comonotonic ones. That behavior changes with increasing SNR. One of the reasons is the steeper slope of the curve of the i.i.d. case for high SNRs. This slope represents the diversity, which is non in the case of independent links. In contrast, the diversity of the comonotonic and worst cases is one.

### E. Zero-Outage Capacity with Perfect CSI-T

We keep the assumption from the previous section that $|h_i|^2 \sim \exp(\lambda)$, but now also assume that the transmitter has perfect CSI. In the following, we will only focus on the zero-outage capacity and compare it to the scenario without CSI-T.

1) **Upper Bound:** The upper bound on the zero-outage capacity with perfect CSI-T for Rayleigh fading can be evaluated according to (14) in Theorem 2. Figure 4 shows this upper bound on the zero-outage capacity for an SNR of 5 dB for different $n$. Since there exists no closed-form solution for determining $c_n$, the values are evaluated numerically.

2) **Lower Bound:** As stated in [12, Cor. 1], the lower bound is attained for comonotonic variables $(|h_1|^2, \ldots, |h_n|^2)$. In our case, $|h_i| \sim \exp(\lambda) = F^{-1}(U)$ with $U \sim U([0,1])$ which gives the following solution to the maximum value of the expected value

$$\max \mathbb{E}[|h_i|^2 - \exp(\lambda)] = \mathbb{E}_U - \mathbb{E}_U[1 \left/ \sum_{i=1}^n |h_i|^2 \right.] = \mathbb{E}_U - \mathbb{E}_U[1 \left/ \sum_{i=1}^n F^{-1}(U) \right.]$$

$$= \frac{1}{n} \int_0^1 \frac{\lambda}{\log(1-u)} \, du.$$  \hspace{1cm} (25)

Since this grows to infinity, the zero-outage capacity from (13) goes down to zero in this worst-case scenario of comonotonic fading coefficients, i.e., $R^0_{\text{CSIT}} = 0$.  

3) **Independent Coefficients:** The sum of independent exponentially distributed random variables $X_i \sim \exp(\lambda)$ follows a Gamma-distribution $\sum_{i=1}^n X_i \sim \Gamma(n, \lambda^{-1})$ (shape and scale) [20, Ch. 17.2]. In our case, we need the expected value of the inverse of such a Gamma-distributed variable. This is distributed according to an inverse-gamma distribution $\text{IG}(n, \lambda^{-1})$ which has the expected value of [20, Ch. 22.4]

$$\mathbb{E}_{\text{IG}(n, \lambda^{-1})} \left[ \frac{1}{U} \right] = \frac{\lambda}{n-1}.$$  \hspace{1cm} (26)

Combining this expression with (13) gives the zero-outage capacity with perfect CSI-T for independent Rayleigh fading coefficients $|h_i| \sim \exp(\lambda)$ as

$$R^0_{\text{CSIT}, \text{id}} = \log_2 \left( 1 + \frac{\rho(n-1)}{\lambda} \right).$$  \hspace{1cm} (27)

4) **Summary:** Figure 4 shows the best-case and i.i.d. case zero-outage capacities with perfect CSI-T for varying $n$ and $|h_i|^2 \sim \exp(1)$. For comparison, the best-case curve from the scenario without CSI-T is also given. Note that the zero-outage for i.i.d. coefficients is always zero in this case.

![Figure 4. Comparison of the best-case and iid case zero-outage capacity with and without perfect CSI at the transmitter for $|h_i|^2 \sim \exp(1)$ and $\rho = 5$ dB.](https://doi.org/10.24355/dbbs.084-202301111337-0)
the fading coefficients. In contrast, this is not true for perfect CSI-T, where the zero-outage capacity is positive, e.g., in the i.i.d. case.

IV. CONCLUSION

In this work, lower and upper bounds on the $\varepsilon$-outage capacity including the zero-outage capacity of fading channels with the same monotone marginal density are provided. Since the individual fading coefficients were not assumed to be independent, these bounds hold for the general case of arbitrary dependency between the fading channels. An extension of this work which covers fading channels with different monotone marginal densities can be found in [12].

A remarkable result is that the zero-outage capacity of $n \geq 2$ Rayleigh fading links without instantaneous CSI-T can be greater than zero. However, a particular dependence structure between the fading coefficients is required. In the worst-case (and even the i.i.d.) case, the zero-outage capacity is equal to zero. Note that one challenge follows from the fact that only proves the existence of such a dependence structure. Deriving an explicit form remains an open problem, not to speak about the ‘practical construction’ of the corresponding propagation scenario.

For practical applications, a finite blocklength is required. Especially in the context of URLLC, short blocklengths are needed to ensure a low latency. Therefore, it will be interesting for future work to extend the results to finite blocklengths, e.g., by using Polyanisky’s work on finite blocklength communication [23, 24].

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